PHYS 320 ANALYTICAL MECHANICS

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Fall 2018

Final Exam time change:
Wednesday, 12 Dec, 6:30 – 8:20 pm

Lagrangian Mechanics

Lagrangian function:

$$L \equiv T(q_j, \dot{q}_j, t) - U(q_j, t)$$

Euler-Lagrange equations of motion:

$$\frac{\partial L}{\partial q_j} \equiv \frac{d}{dt}\frac{\partial L}{\partial \dot{q}_j}, \quad j = 1, 2, 3, \ldots, s$$

Lagrangian Mechanics:

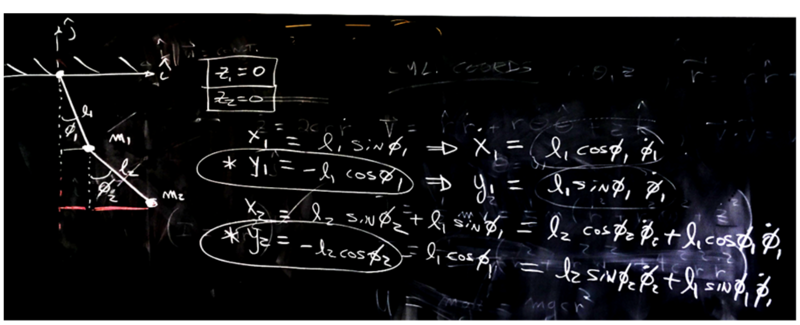
$$\frac{\partial L}{\partial q_j} \equiv \text{ generalized force, jth component}$$

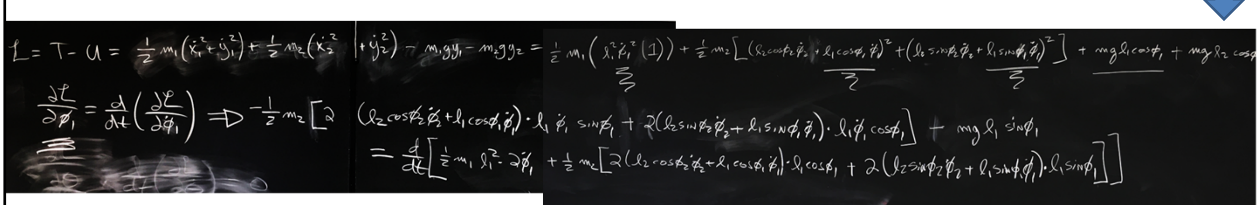
$$\frac{\partial L}{\partial \dot{q}_j} = \text{ generalized momentum, jth component}$$

$$\frac{\partial L}{\partial q_j} \equiv \frac{d}{dt}\frac{\partial L}{\partial \dot{q}_j}$$

generalized force
= rate of change of
generalized momentum

Double Pendulum





Double Pendulum

After a bit of algebra …

$$L = \frac{1}{2}\left[m_1 l_1^2 \dot{\phi}_1^2 + m_2 l_1^2 \dot{\phi}_1^2 + m_2 l_2^2 \dot{\phi}_2^2 + 2 m_2 l_1 l_2 \dot{\phi}_1 \dot{\phi}_2 \cos(\phi_1 - \phi_2) \right] + g\left[m_1 l_1 \cos\phi_1 + m_2 l_1 \cos\phi_1 + m_2 l_2 \cos\phi_2 \right]$$

The equations of motion are found from $\frac{\partial L}{\partial \phi_1} \equiv \frac{d}{dt}\frac{\partial L}{\partial \dot{\phi}_1}$ and $\frac{\partial L}{\partial \phi_2} \equiv \frac{d}{dt}\frac{\partial L}{\partial \dot{\phi}_2}$

Let's explore the case where $m_1 = m_2$ and $l_1 = l_2$:

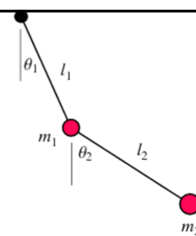
$$L = \frac{ml^2}{2}\left[\dot{\phi}_1^2 + \dot{\phi}_1^2 + \dot{\phi}_2^2 + 2\dot{\phi}_1 \dot{\phi}_2 \cos(\phi_1 - \phi_2) \right] + gml\left[2\cos\phi_1 + \cos\phi_2 \right]$$

The equations of motion are then

$$2\ddot{\phi}_1 + \ddot{\phi}_2 \cos(\phi_1 - \phi_2) + \dot{\phi}_2^2 \sin(\phi_1 - \phi_2) + \frac{2g}{l}\sin\phi_1 = 0$$

$$\ddot{\phi}_2 + \ddot{\phi}_1 \cos(\phi_1 - \phi_2) - \dot{\phi}_1^2 \sin(\phi_1 - \phi_2) + \frac{2g}{l}\sin\phi_2 = 0$$

Double Pendulum



The equations of motion can be put into matrix form:

$$\mathrm{M}\ddot{\Phi} = -\mathrm{K}\Phi$$

$$\Phi = \begin{bmatrix} \phi_1 \\ \phi_2 \end{bmatrix}, \quad \mathrm{M} = \begin{bmatrix} (m_1 + m_2)L_1 & m_2 L_1 L_2 \\ m_2 L_1 L_2 & m_2 L_2^2 \end{bmatrix}, \quad \mathrm{K} = \begin{bmatrix} (m_1 + m_2)gL_1 & 0 \\ 0 & m_2 g L_2 \end{bmatrix}$$

The solutions can be expressed in complex form:

$$\Phi = \begin{bmatrix} \phi_1 \\ \phi_2 \end{bmatrix} = \mathrm{Re}\{\mathrm{Z}(t)\} = \mathrm{Re}\left\{\begin{bmatrix} z_1 \\ z_2 \end{bmatrix}\right\} = \mathrm{Re}\left\{ a e^{i\omega t} \right\} = \mathrm{Re}\left\{ \begin{bmatrix} a_1 \\ a_2 \end{bmatrix} e^{i\omega t} \right\} = \mathrm{Re}\left\{ \begin{bmatrix} \alpha_1 e^{i\delta_1} \\ \alpha_2 e^{i\delta_2} \end{bmatrix} e^{i\omega t} \right\}$$

As a result, we get an eigenvalue equation; solutions exist if the determinant of the characteristic *equation* is zero:

$$-\omega^2 \mathrm{M} a e^{i\omega t} = -\mathrm{K} a e^{i\omega t}$$

$$\left[\mathrm{K} - \omega^2 \mathrm{M} \right] a = 0$$

$$\Rightarrow \text{ if } \left| \mathrm{K} - \omega^2 \mathrm{M} \right| = 0$$

Lagrangian Mechanics: Extracting Forces
Lagrange multipliers

Holonomic equations of constraint: $f(q_j, t) = 0$

Generalized forces: $Q_k \equiv \lambda_j \dfrac{\partial f}{\partial q_j}, \quad j = 1, 2, 3, \ldots, s$

Taylor: F_k^{cstr}

The Euler-Lagrange Equation becomes:

$$\frac{\partial L}{\partial q_j} + \sum_k \lambda_k \frac{\partial f}{\partial q_j} = \frac{d}{dt}\frac{\partial L}{\partial \dot{q}_j}, \quad \begin{cases} j = 1, 2, 3, \ldots, s \\ k = 1, 2, 3, \ldots, m \end{cases}$$

Hamiltonian Mechanics

Hamiltonian function:

$$H \equiv \sum_j p_j \dot{q}_j - L(q_k, \dot{q}_k, t) \quad \text{where} \quad p_j \equiv \frac{\partial L}{\partial \dot{q}_j}$$

$(H \equiv T + U)$

for conservative systems

Hamilton's equations of motion:

$$\dot{p}_k \equiv -\frac{\partial H}{\partial q_k} \quad \text{and} \quad \dot{q}_k \equiv \frac{\partial H}{\partial p_k}, \quad k = 1, 2, 3, \ldots, s$$

2s 1st order differential equations
(c.p. Lagrange method: n 2^{nd} order DEs)